A Optimization technique of Hydrothermal Systems using Calculus of Variations

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Abstract. This paper falls within the scope of studies concerning the optimization of the functioning of systems with renewable energy (hydro energy). We have developed a theory that is extraordinarily simpler than previous ones which resolves the problem of minimization of a functional of the type

$$F(z)=\int_0^T L(t,z(t),z'(t))dt$$

within the set KC^1 (piecewise C^1) functions that satisfy: $z(0)=0$, $z(T)=b$ and the constraints

$$z'(t)\geq 0 \text{ and } H(t,z(t),z'(t))\leq P_d(t),\ \forall t\in[0,T]$$

In particular, we have established a necessary condition for the stationary functions of the functional. The method allows for elaboration of the optimization algorithm which provides us with the optimal regime of functioning of the entire hydrothermal system. Finally, we present a example, employing the algorithm realized with the "Mathematica" package to this end.

Key Words

Optimization, Hydrothermal System, Calculus of Variations, Hydro Energy, Cost Functional.

1. Introduction

A hydrothermal system is made up of hydraulic and thermal power plants which, during a definite time interval, must jointly satisfy a certain demand in electric power. Thermal plants generate power at the expense of fuel consumption (which is the object of minimization), while hydraulic plants obtain power from the energy liberated by water that moves a turbine; there is a limited quantity of water available during the optimization period.

In prior studies [1-2], it has been proven that the problem of optimization of the fuel costs of a hydrothermal system with n thermal power plants may be reduced to the study of a hydrothermal system made up of one single thermal power plant, called the thermal equivalent. In the present paper, we consider a hydrothermal system with one hydraulic power plant and n thermal power plants that have been substituted by their thermal equivalent. With these conditions, we present the problem from the Electrical Engineering, perspective to then go on to resolve the mathematical problem thus formulated.

A. Hydrothermal Statement of the Problem

The problem consists in minimizing the cost of fuel needed to satisfy a certain power demand during the optimization interval $[0,T]$. Said cost may be represented by the functional

$$F(P)=\int_0^T \Psi(P(t))dt$$

where Ψ is the function of thermal cost of the thermal equivalent and $P(t)$ is the power generated by said plant. Moreover, the following equilibrium equation of active power will have to be fulfilled

$$P(t)+H(t,z(t),z'(t))=P_d(t),\ \forall t\in[0,T]$$

where $P_d(t)$ is the power demand and $H(t,z(t),z'(t))$ is the power contributed to the system at the instant t by the hydraulic plant, being: $z(t)$ the volume that is discharged up to the instant t (in what follows, simply volume) by the plant, and $z'(t)$ the rate of water discharge at the instant t of the plant.

Taking into account the equilibrium equation, the problem reduces to calculating the minimum of the functional

$$F(z)=\int_0^T \Psi\left(P_d(t)-H\left(t,z(t),z'(t)\right)\right)dt$$

If we assume that b is the volume of water that must be discharged during the entire optimization interval, the following boundary conditions will have to be fulfilled

$$z(0)=0,\ z(T)=b$$

The classic studies dealing with hydrothermal optimization employ concrete models both for the function of thermal cost Ψ, as well as for the function of effective hydraulic generation H, so if the model changes, the results obtained are not valid.

The study of optimal conditions for the functioning of a hydrothermal system constitutes a complicated problem which has attracted significant interest in recent decades. Various techniques have been applied to solve the problem, such as functional analysis techniques [3], dynamic programming [4], Ritz's method [5], network techniques [6] and others [7-8].

Such a variety of the mathematical models forces us to undertake a general study of the problem. The results of this study should be extensible to a large set of the hydrothermal problems. It is worth observing that most of the studies on the subject are somewhat indefinite in determining the limits of applicability of their results and hence, a thorough theoretical study is needed, one that goes beyond a mere analysis of separate problems.

One of the main contributions of this paper is that the method is valid for any model of power plants, since we will try to consider the functions P_d, Ψ and H as general as possible without any restrictions, except those that are natural for problems of this type. For the sake of convenience, we assume throughout the paper that they are sufficiently smooth and are subject to the following additional assumptions:

Function of thermal cost. Let us assume that the cost function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies $\Psi'(x) > 0$, $\forall x \in \mathbb{R}^+$ and, thus, is strictly increasing. This restriction is absolutely natural: it reads more cost to more generated power. Let us assume as well that $\Psi''(x) > 0$, $\forall x \in \mathbb{R}^+$ and is therefore strictly convex. The models traditionally employed meet this restriction.

Function of effective hydraulic generation. Let us assume that the hydraulic generation $H(t, z, z')$: $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is strictly increasing with respect to the rate of water discharge z', with $H_{z'} > 0$. Let us also assume that $H(t, z, z')$ is concave with respect to z', that is $H_{z'z'} \leq 0$. The real models meet these two restrictions, and the former means more power to a higher rate of water discharge. We also suppose that $H(t, z, 0) = 0$.

We see that we only admit non-negative thermal power $(P(t))$ and we will solely admit non-negative volumes $(z(t))$ and rates of water discharge $(z'(t))$, therefore we may present the mathematical problem in the following terms.

B. Variational Statement of the Problem

We will call Π_b the problem of minimization of the functional

$$F(z) = \int_0^T L(t, z(t), z'(t))dt$$

with L of the form

$$L(t, z(t), z'(t)) = \Psi(P_d(t) - H(t, z(t), z'(t)))$$

over the set

$$\Theta_b = \{z \in KC^1[0, T] \;/\; z(0) = 0, z(T) = b, \\ z'(t) \geq 0 \wedge H(t, z(t), z'(t)) \leq P_d(t)\}$$

So the problem involves inequality non-holonomic constraints in the derivative $z'(t)$.

Variational problems in which the derivatives of the admissible functions must be subject to certain inequality constraints (differential inclusion $z' \in E(t, z)$) have traditionally been dealt with by recurring to many diverses techniques. The first studies in this field were conducted by Flodin [9] for simpler constraints of the type $A \leq z'(t) \leq B$ and by Follinger, who in [10] deals in a very complex way with a more general constraint of the type $H(t, z(t)) \leq z'(t) \leq G(t, z(t))$. In [11] Clarke deals with necessary conditions for problems in the calculus of variations that incorporate inequality constraints of the form $f(z, z') \leq 0$. In [12], the author determines necessary conditions, in terms of generalized gradients, for the existence of an extremal arc for calculus of variations and optimal control problems with differential multi-inclusion $z' \in E(t, z)$.

In [13] Clarke and Loewen consider an optimal control problem on a fixed time interval $[0, T]$, and a variety of necessary conditions are derived for the original optimal control problem. The same authors, in [14], develop an existence theory for solutions to the original problem with $|z'(t)| < R$. In [15-16] Loewen and Rockafellar consider the classical Bolza problem in the calculus of variations, incorporating endpoint and velocity constraints through infinite penalties. The integrand L are allowed to be nondifferentiable. In [17] the authors have recurred to techniques of optimal control and they formulate a sufficient optimality condition for broken extremals in terms of the solution of the Hamilton-Jacobi-Bellman equation.

In [18-19-20] the simplest problem of the calculus of variations is investigated, along with the corresponding Euler equation. Some new results on the Euler equation are obtained. Finally, Noble and Schättler [21] develop sufficient conditions for a relative minimum for broken extremals in an optimal control problem based on the method of characteristics.

In this paper, we have developed a much simpler theory than previous ones which solves the problem Π_b. The development is hence self-contained and extremely basic, and also enables the construction of the optimal solution.

2. Boundary Solutions

We will say that a function q is admissible for Π_b if $q \in \Theta_b$. We will say that q is a solution of problem Π_b if q is admissible and

$$F(q) \leq F(z), \; \forall z \in \Theta_b$$

We also say that a function $q \in \Theta_b$ presents an inferior boundary arc in $[t_1, t_2]$ if

$$\forall t \in [t_1, t_2], \; q'(t) = 0$$

and that a function $q \in \Theta_b$ presents a superior boundary arc in $[t_1, t_2]$ if

$$\forall t \in [t_1, t_2],\ H(t, q(t), q'(t)) = P_d(t)$$

The function $q \in \Theta_b$ presents an interior (or extremal) arc in an interval $[t_1, t_2]$ if

$$\forall t \in (t_1, t_2),\ H(t, q(t), q'(t)) < P_d(t) \wedge 0 < q'(t)$$

If we did not have the restrictions $z'(t) \geq 0$ and $H(t, z(t), z'(t)) \leq P_d(t)$, we could use the shooting method to resolve the problem. In this case, we would use the integral form of the Euler's equation (Du Bois-Reymond equation)

$$\begin{aligned} &- L_{z'}(t, z(t), z'(t)) + \int_0^t L_z(s, z(s), z'(s))ds = \\ &= -L_{z'}(0, z(0), z'(0)) = K > 0,\ \forall t \in [0, T] \quad (1) \end{aligned}$$

Varying the initial condition of the derivative $z'(0)$ (initial flow rate), we would search for the extremal that fulfils the second boundary condition $z(T) = b$ (final volume). However, we cannot use this method in our case, as due to the restrictions, the extremals may not admit bilateral variations, i.e. they may present boundary arcs. The following questions arise: Do all the interior arcs (C_1 and C_3 in Fig. 1) have the same constant K?. At what moments does the boundary have to be penetrated and abandoned?. In the following sections, we shall develop the theory needed to respond to these questions.

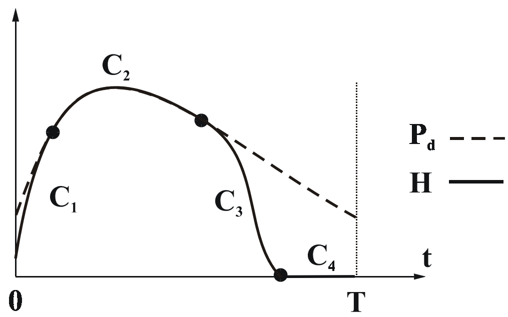

Fig. 1. Boundary and interior arcs

3. The Main Coordination Theorem

Firstly, we are going to introduce the concept of weak influence of the volume, essential when studying superior boundary arcs.

Definition 1. We say that in a problem Π_b, the influence of the volume is weak if for all admissible q, $\forall h \in KC^1[0, T]$ and $\forall t_1, t_2 \in [0, T]$ satisfying

i) $H(t, q(t), q'(t)) = P_d(t)$ in $[t_1, t_2]$
ii) $h(0) = h(T) = 0$
iii) $h'(t) < 0,\ \forall t \in (t_1, t_2)$

$\exists \varepsilon' > 0$ such that $\forall \varepsilon \in [0, \varepsilon']$ and $\forall t \in [t_1, t_2]$

$$H(t, q(t) + \varepsilon h(t), q'(t) + \varepsilon h'(t)) \leq P_d(t)$$

This definition in fact serves to impose the restriction that $q + \varepsilon h$ is admissible $\forall \varepsilon \in [0, \varepsilon']$ and for any h which satisfies ii) and iii). This yields the existence of

$$\delta F(q; h) := \lim_{x \to 0^+} \frac{F(q + xh) - F(q)}{x}$$

The definition is designed to reflect the fact that if the hydraulic power station generates all the power demand in an interval, any other distribution of water that produces a lower rate of water discharge in the same interval is unable to produce more power at any instant of the interval, or what is equivalent, the flow rate has a much greater influence on the generation of hydraulic power than the volume, a point that all the hydraulic power plant models in use fulfill.

Let us next see the results that give rise to what we have denominated the main coordination theorem, which will enable us to find the optimum solution.

Definition 2. Let us term the coordination function of $q \in \Theta_b$ the function in $[0, T]$, defined as follows

$$¥_q(x) = \int_0^x L_z(t, q(t), q'(t))dt - L_{z'}(x, q(x), q'(x))$$

We observe that for every $t_0, t_1 \in [0, T]$ we have

$$\begin{aligned} ¥_q(t_1) - ¥_q(t_0) = &\int_{t_0}^{t_1} L_z(s, q(s), q'(s))ds + \\ &+ L_{z'}(t_0, q(t_0), q'(t_0)) - L_{z'}(t_1, q(t_1), q'(t_1)) \end{aligned}$$

Theorem 1. Let $\epsilon > 0$ and $q \in C^1$ be a solution of problem Π_b such that

$$q'(t) = 0,\ \forall t \in [t_0, t_1]$$

i) If $0 < H(t, q(t), q'(t)) < P_d(t),\ \forall t \in [t_1, t_1 + \epsilon]$, then

$$¥_q(t_1) \geq ¥_q(t_0)$$

ii) If $0 < H(t, q(t), q'(t)) < P_d(t),\ \forall t \in [t_0 - \epsilon, t_0]$, then

$$¥_q(t_1) \leq ¥_q(t_0)$$

Proof)

i) Let us assume the contrary, that is

$$¥_q(t_1) < ¥_q(t_0)$$

or, which is the same

$$\int_{t_0}^{t_1} \left(L_z(t, q(t), q'(t)) - \frac{d}{dt} L_{z'}(t, q(t), q'(t)) \right) dt < 0$$

Let us consider the following sequence of functions

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t \leq t_0 \\ n(t - t_0) & \text{if} \quad t \in [t_0, t_0 + \frac{1}{n}] \\ 1 & \text{if} \quad t \in [t_0 + \frac{1}{n}, t_1] \\ g(t) & \text{if} \quad t > t_1 \end{cases}$$

with $g \in C^1$ such that $g(t_1) = 1$ and $g(t) = 0$ $\forall t \in [t_1 + \epsilon, T]$.

It is clear that f_n point-wise convergent and uniformly bounded in $[0, T]$, and integrable in the sense of Riemann $\forall n$. Moreover

$$\lim_{n\to\infty} f_n(t) = f(t) = \begin{cases} 0 & \text{if} \quad t \le t_0 \\ 1 & \text{if} \quad t \in (t_0, t_1] \\ g(t) & \text{if} \quad t > t_1 \end{cases}$$

Letting

$$\Gamma(t) = \left(L_z(t, q(t), q'(t)) - \frac{d}{dt} L_{z'}(t, q(t), q'(t)) \right)$$

and

$$\Upsilon_n(t) = \Gamma(t) f_n(t)$$

we have that $\Upsilon_n(t)$ also is point-wise convergent and uniformly bounded in $[0, T]$ and integrable in the sense of Riemann $\forall n$. Besides, there hold the relations

$$\lim_{n\to\infty} \Upsilon_n(t) = \begin{cases} 0 & \text{if} \quad t \le t_0 \\ \Gamma(t) & \text{if} \quad t \in (t_0, t_1] \\ \Gamma(t) g(t) & \text{if} \quad t > t_1 \end{cases}$$

We are now in condition to establish that

$$\lim_{n\to\infty} \int_{t_0}^{t_1} \Upsilon_n(t) dt = \int_{t_0}^{t_1} \lim_{n\to\infty} \Upsilon_n(t) dt = \int_{t_0}^{t_1} \Gamma(t) dt$$

Thus

$$\lim_{n\to\infty} \int_{t_0}^{t_1} \Upsilon_n(t) dt =$$

$$= \int_{t_0}^{t_1} \left(L_z(t, q(t), q'(t)) - \frac{d}{dt} L_{z'}(t, q(t), q'(t)) \right) dt < 0$$

which yields that for some m

$$\int_{t_0}^{t_1} \Upsilon_m(t) dt = \int_{t_0}^{t_1} \Gamma(t) f_m(t) dt < 0$$

Let us take into account that $\exists x' > 0$ so small that, $\forall x \in [0, x']$, $q(t) + x f_m(t)$ is admissible. At this moment it is not necessary to claim the weakness of the influence of the volume. This is because the restriction $q'(t) \ge 0$ only effects on the rate of water discharge. Let us consider

$$\begin{aligned} \delta F(q; f_m) &= \lim_{x\to 0^+} \frac{F(q + x f_m) - F(q)}{x} \\ &= \int_0^T \left(L_z(t, q, q') - \frac{d}{dt} L_{z'}(t, q, q') \right) \cdot f_m(t) dt \\ &= \int_0^{t_0} \Gamma(t) f_m(t) dt + \int_{t_0}^{t_1} \Gamma(t) f_m(t) dt + \\ &\quad + \int_{t_1}^{t_1+\epsilon} \Gamma(t) g(t) dt + \int_{t_1+\epsilon}^{T} \Gamma(t) \cdot f_m(t) dt \end{aligned}$$

Since $\Gamma(t) = 0$ in $[t_1, t_1 + \epsilon]$, q is an extremal in this interval, and $f_m \equiv 0$ in $[t_1 + \epsilon, T] \cup [0, t_0]$, it follows that

$$\begin{aligned} \delta F(q; f_m) &= \lim_{x\to 0^+} \frac{F(q + x f_m) - F(q)}{x} \\ &= \int_{t_0}^{t_1} \Gamma(t) f_m(t) dt < 0 \end{aligned}$$

which contradicts the assumption that $q(t)$ minimizes the functional.

ii) We proceeding by analogy and consider the auxiliary sequence of functions

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t \ge t_1 \\ n(t - t_1) & \text{if} \quad t \in [t_1 - \frac{1}{n}, t_1] \\ -1 & \text{if} \quad t \in [t_0, t_1 - \frac{1}{n}] \\ g(t) & \text{if} \quad t \le t_0 \end{cases}$$

with $g \in C^1$ such that $g(t_0) = -1$ and $g(t) = 0$ if $t \le t_0 - \epsilon$. □

Theorem 2. Let $\epsilon > 0$ and $q \in C^1$ be an solution of problem Π_b, with the weak influence of the volume, such that

$$H(t, q(t), q'(t)) = P_d(t), \ \forall t \in [t_0, t_1]$$

i) If $0 < H(t, q(t), q'(t)) < P_d(t)$, $\forall t \in [t_1, t_1 + \epsilon]$, then

$$¥_q(t_1) \ge ¥_q(t_0)$$

ii) If $0 < H(t, q(t), q'(t)) < P_d(t)$, $\forall t \in [t_0 - \epsilon, t_0]$, then

$$¥_q(t_1) \le ¥_q(t_0)$$

Proof)

The proof is technically identical to the one given in the previous theorem. They only differ in the choice of the auxiliary sequences of functions and in the way of using the weakness of influence of the volume in the calculation of the "directional derivative"; now the admissibility of $q(t) + x f_m(t)$ is required.

i) One can use the following sequence of auxiliary functions, satisfying the condition $f_n'(t) < 0$, $\forall t \in (t_0, t_1)$

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t \le t_0 \\ \left(\frac{t_1 - t}{t_1 - t_0} \right)^n - 1 & \text{if} \quad t \in [t_0, t_1] \\ g(t) & \text{if} \quad t > t_1 \end{cases}$$

with $g \in C^1$ such that $g(t_1) = -1$ and $g(t) = 0$ if $t \ge t_1 + \epsilon$.

ii) In this case one can use the sequence of auxiliary functions, such that $f_n'(t) < 0$, $\forall t \in (t_0, t_1)$, defined by

$$f_n(t) = \begin{cases} 0 & \text{if} \quad t > t_1 \\ \left(\frac{t_1 - t}{t_1 - t_0} \right)^{\frac{1}{n}} & \text{if} \quad t \in [t_0, t_1] \\ g(t) & \text{if} \quad t < t_0 \end{cases}$$

with $g \in C^1$ such that $g(t_0) = 1$ and $g(t) = 0$ if $t \le t_0 - \epsilon$. □

Corollary 1. If $q \in C^1$ is a solution of problem Π_b, which contains in $[t_0, t_1]$ a boundary arc of the type $q'(t) = 0$, and in both $[t_0 - \epsilon, t_0]$ and $[t_1, t_1 + \epsilon]$ interior arcs of the extremal, then the following is true

$$¥_q(t_1) = ¥_q(t_0) \quad (2)$$

$$¥_q(t) = const., \forall t \in [t_0 - \epsilon, t_0] \cup [t_1, t_1 + \epsilon] \quad (3)$$

Proof)

(2) An immediate consequence of the above theorem.

(3) In the intervals $[t_0 - \epsilon, t_0]$ and $[t_1, t_1 + \epsilon]$, $q(t)$ satisfies (1)

$$\begin{aligned}
K_0 &= \int_0^t L_z(s, q(s), q'(s))ds - L_{z'}(t, q(t), q'(t)) \\
&\quad \text{in } [t_0 - \epsilon, t_0] \\
K_0 &= \int_0^{t_0} L_z(s, q(s), q'(s))ds - L_{z'}(t_0, q(t_0), q'(t_0)) \\
K_1 &= \int_0^t L_z(s, q(s), q'(s))ds - L_{z'}(t, q(t), q'(t)) \\
&\quad \text{in } [t_1, t_1 + \epsilon] \\
K_1 &= \int_0^{t_1} L_z(s, q(s), q'(s))ds - L_{z'}(t_1, q(t_1), q'(t_1))
\end{aligned}$$

We have that

$$K_1 - K_0 = ¥_q(t_1) - ¥_q(t_0) = 0$$

and hence $K_1 = K_0$. □

Corollary 2. If $q \in C^1$ is a solution of problem Π_b, with the weak influence of the volume, containing the boundary arc of the type

$$H(t, q(t), q'(t)) = P_d(t), \ \forall t \in [t_0, t_1]$$

and interior extremal arcs in both $[t_0 - \epsilon, t_0]$ and $[t_1, t_1 + \epsilon]$, then the following holds: (2) and (3).

Proof)

The proof is similar to the previous corollary. □

These two corollaries give the satisfactory answer to the above-formulated question: the constant K is the same for two different arcs of extremal.

Theorem 3. (The main coordination theorem). If $q \in C^1$ is a solution of problem Π_b, with the weak influence of the volume, then $\exists K$ such that

$$\text{i) If } \left\{ \begin{array}{c} q'(t) > 0 \\ \text{and} \\ H(t, q(t), q'(t)) < P_d(t) \end{array} \right\} \Longrightarrow ¥_q(t) = K$$
$$\text{ii) If } q'(t) = 0 \Longrightarrow ¥_q(t) \leq K$$
$$\text{iii) If } H(t, q(t), q'(t)) = P_d(t) \Longrightarrow ¥_q(t) \geq K$$

Proof)

i) This was already proven for the consequent interior arcs in the previous corollaries.

ii) If $q'(t) = 0$, $\forall t \in [t_0, t_1]$, being $(t_1, t_1 + \epsilon)$ an interior arc, by virtue of continuity of $¥_q$ we have that

$$¥_q(t_1) = K$$

Applying Theorem 1, we conclude that

$$¥_q(t_1) - ¥_q(t) \geq 0, \forall t \in [t_0, t_1]$$

and, therefore,

$$K = ¥_q(t_1) \geq ¥_q(t)$$

iii) By analogy with the previous argument. □

The constant K will be termed the *coordination constant* of the solution q.

4. Construction of the Optimal Solution

We have already mentioned the fact that if we did not have inequality restrictions, the solution could be constructed by means of the shooting method. We use the same framework in the present case, but the variation of the initial condition for the derivative, which now need not make sense, is substituted by the variation of the coordination constant K.

The problem will consist in finding for each K the function q_K which satisfies $q_K = 0$ and the conditions of the main coordination theorem, and from among these functions, the one which generates an admissible function ($q_K(T) = b$).

We will denote by M the rate of water discharge at the instant $t = 0$ that is necessary for the hydraulic power station to satisfy the power demand: $H(0, 0, M) = P_d(0)$ and we will denote by m the rate of water discharge at the instant $t = 0$ that is necessary for $H(0, 0, m) = 0$. We also set

$$K_m = -L_{z'}(0, 0, m); \quad K_M = -L_{z'}(0, 0, M)$$

We observe that $\forall x \in (m, M)$ (with the hypothesis $L_{z'z'}(t, z, z') > 0$) we have

$$K_M < -L_{z'}(0, 0, x) < K_m$$

To construct q_K, we proceed by the following steps.

Step 1] (the first arc)

i) If $K \geq K_m$, we set $q_K(t) = 0$, in the maximal interval $[0, t_1]$, where $\forall t \in [0, t_1]$

$$K \geq ¥_0(t) = \int_0^t L_z(s, 0, 0)ds - L_{z'}(t, 0, 0)$$

(The thermal power station generates all the power demanded in $[0, t_1]$).

ii) If $K \leq K_M$, we set $q_K(t) = \omega(t)$, the solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = P_d(t) \text{ with } \omega(0) = 0$$

in the maximal interval $[0, t_1]$, where $\forall t \in [0, t_1]$

$$K \leq \yen_\omega(t) = \\ = \int_0^t L_z(s, \omega(s), \omega'(s))ds - L_{z'}(t, \omega(t), \omega'(t))$$

(The hydraulic power station generates all the power demanded in $[0, t_1]$).

iii) $K_M < K < K_m$ ($\exists x$ such that $K = -L_{z'}(0, 0, x)$).

q_K will be the arc of the interior extremal (with $q_K(0) = 0$) which satisfies Euler's equation in its maximal domain $[0, t_1]$ and therefore the coordination equation, $\forall t \in [0, t_1)$

$$K = \yen_{q_K}(t) = \\ = \int_0^t L_z(s, q_K(s), q'_K(s))ds - L_{z'}(t, q_K(t), q'_K(t))$$

i-th Step] (i-th arc)

A) If q_K has an interior arc in $[t_{i-1}, t_i]$, there are two possibilities:

I) If $q'_K(t_i) = 0$, we consider the maximal interval $[t_i, t_{i+1}]$ such that $\forall t \in [t_i, t_{i+1}]$

$$K \geq \int_0^{t_i} L_z(s, q_K(s), q'_K(s))ds + \\ + \int_{t_i}^t L_z(s, q_K(t_i), 0)ds - L_{z'}(t, q_K(t_i), 0)$$

If this is the case, we set $q_K(t) = q_K(t_i)$, $\forall t \in [t_i, t_{i+1}]$.

II) If $H(t_i, q_K(t_i), q'_K(t_i)) = P_d(t_i)$, we consider the maximal interval $[t_i, t_{i+1}]$ such that $\forall t \in [t_i, t_{i+1}]$

$$K \leq \int_0^{t_i} L_z(s, q_K(s), q'_K(s))ds + \\ + \int_{t_i}^t L_z(s, \omega(s), \omega'(s))ds - L_{z'}(t, \omega(t), \omega'(t))$$

$\omega(t)$ being a solution of the differential equation

$$H(t, \omega(t), \omega'(t)) = P_d(t) \text{ with } \omega(t_i) = q_K(t_i)$$

If this is the case, we set $q_K(t) = \omega(t)$, $\forall t \in [t_i, t_{i+1}]$.

B) If $[t_{i-1}, t_i]$ is the boundary interval, we consider the maximal interval $[t_i, t_{i+1}]$ such that $\forall t \in [t_i, t_{i+1}]$

$$K = \int_0^{t_i} L_z(s, q_K(s), q'_K(s))ds + \\ + \int_{t_i}^t L_z(s, \omega(s), \omega'(s))ds - L_{z'}(t, \omega(t), \omega'(t)) \quad (4)$$

$\omega(t)$ being an interior arc of the extremal, with $\omega(t_i) = q_K(t_i)$, which satisfies Euler's equation in its maximal domain $[t_i, t_{i+1}]$ and therefore satisfies the coordination equation (4). Now, we set $q_K(t) = \omega(t)$, $\forall t \in [t_i, t_{i+1}]$.

From the computational point of view, the construction of q_K can be performed with the same procedure as in the shooting method, with the use of a discretized version of equation (1). The exception is that at the instant when the values obtained for z and z' do not obey the restrictions, we force the solution q_K to belong to the boundary until the moment when the conditions of leaving the domain (established in the main coordination theorem) are fulfilled.

5. A Numerical Example

A program was elaborated using the Mathematica package which resolves the optimization problem and was then applied to a hydrothermal system made up of the thermal equivalent and a hydraulic plant.

For the fuel cost model of the equivalent thermal plant, we use the quadratic model

$$\Psi(P(t)) = \alpha + \beta P(t) + \gamma P(t)^2$$

The units for the coefficients are: α in (€/h); β in (€/$h.Mw$); γ in (€/$h.MW^2$).

The hydro-plant's active power generation is given by

$$P_h(t) = -A(t)z'(t) - Bz'(t)z(t) - Cz'(t)^2$$

where the coefficients A, B and C are

$$A(t) = \frac{-1}{G} B_y(S_0 + t \cdot i), \quad B = \frac{B_y}{G}, \quad C = \frac{B_T}{G}$$

We consider that the transmission losses for the hydro-plant are expressed by Kirchmayer's model, with the following loss equation: $b_l \cdot (P_h(t))^2$. So,

$$H(t) = P_h(t) - b_l \cdot (P_h(t))^2$$

The units for the coefficients of the hydro-plant are: the efficiency G in $(m^4/h.Mw)$, the restriction on the volume b in (m^3), the loss coefficient b_l in $(1/Mw)$, the natural inflow i in (m^3/h), the initial volume S_0 in (m^3), the coefficients B_T in $(m^{-2}.h)$ and the coefficients B_y in (m^{-2}) (parameters that depend on the geometry of the tanks).

The data for the thermal and hydraulic plants are summarized in Table I.

TABLE I.- Coefficients

α	β	γ	G	i
0	4	0.001	526315	10190000

S_0	B_T	B_y	b_l
$200 \cdot 10^9$	$581.740 \cdot 10^{-10}$	$149.5 \cdot 10^{-12}$	0.0002

The values of the power demand (in Mw) were adjusted to the following curve

$$P_d(t) = 1003 + 6\,\mathrm{sen}[\frac{4\pi t}{24}] - 3t(24-t)2\cos[\frac{4\pi t}{24}]$$

Firstly, an optimization interval of 24 h. was considered, and a final volume $b = 30 \cdot 10^6\ m^3$.

Fig. 2 presents the plots of power demand (P_d), thermal power (P) and effective hydraulic power (H). We can see that from 9 h. until 15 h., corresponding to the hours of the lowest demand in power (i.e. with the most pronounced trough), the hydraulic plant stops functioning and the thermal plant assumes all the power demand. This is done to reserve water for when power demands are very high, which corresponds to the peaks that can be seen in the figure. In this case, the cost is 120848 €.

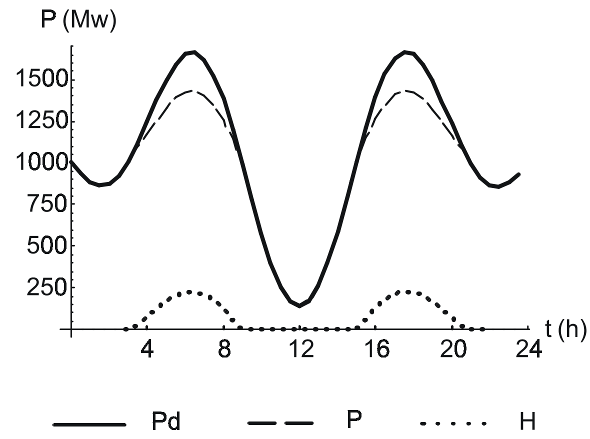

Fig. 2. Optimal solution with $b = 30 \cdot 10^6 m^3$

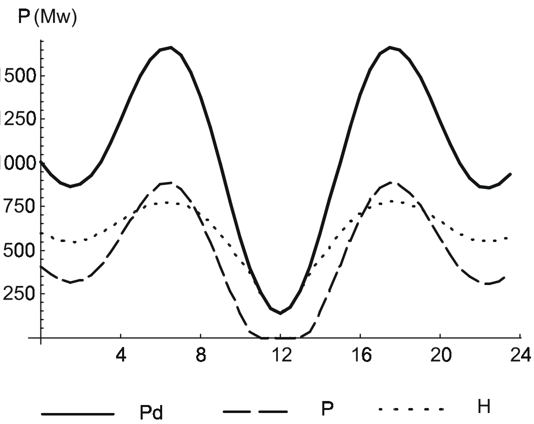

Fig. 3. Optimal solution with $b = 300 \cdot 10^6 m^3$

However, if we take a larger final volume, $b = 300 \cdot 10^6\ m^3$, the solution is that depicted in Fig. 3. Here we see that as there is sufficient water, the hydraulic plant does not stop functioning at any time whatsoever, though the thermal plant shuts off in the most pronounced trough, i.e. from 11 h. until 13 h. In this case, the fuel cost is 51265.50 €, which logically is considerably lower.

6. Conclusions

This paper falls within the scope of studies concerning the optimization of the functioning of systems with renewable energy (hydro energy).

From the Engineering perspective, one of the main contributions of this paper is that it resolves the optimization of hydrothermal systems without being restricted to particular cases. That is to say, the studied carried out is independent of of the models used both for thermal and for hydraulic power plants, in contrast to the majority of studies in this field, which use concrete models. What is more, we have obtained a very simple method that enables us to find an optimal solution in the presence of inequality constraints, and which requires very little computational effort.

From the mathematical point of view, we have also obtained notable results. When constraints are not considered, the determination of the extremals of a functional is a very simple problem and its solution simply consists in solving the Euler-Lagrange equation of the functional with boundary conditions, for which the shooting method can be employed. However, the problem that we have resolved in this paper is more complicated, due to non-holonomic constraints.

The main contribution of this paper is the characterization of the extremals in variational problems with non-holonomic constraints. Said characterization, set out in Theorem 3, permits the solution to be constructed by means of a method inspired by the shooting method that is much simpler than those employed until now for resolving this type of problem.

As far as future perspectives are concerned, it would be most interesting to apply this method when the system is made up of n hydraulic power plants.

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